Summary: Itô's integrals,
Itô's diffusions, Itô's formula

Motivation: In discrete time, when we use the binomial model, we assume a specific dynamics for the stock price.

discrete time
$$\begin{cases} S_1 = S_0 Z_1 & Z_1 = \begin{cases} u & \text{with prob. } p \\ d & \text{" " } 1-p \end{cases} \\ S_2 = S_0 Z_1 Z_2 & Z_2 \overset{i.i.d.}{\sim} Z_1 \ldots \\ S_n = S_0 \prod_{i=1}^{n} Z_i & Z_i \overset{iid}{\sim} Z = \begin{cases} u & \text{with } p \\ d & \text{" } 1-p \end{cases} \end{cases}$$

Question: how do we model stock prices at continuous time?

Under the Black-Scholes model: $\{S(t),\ t \in (0,T]\}$
such that

$$\boxed{dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t)} \quad (*)$$

$$\Leftrightarrow$$

$$\boxed{\frac{dS(t)}{S(t)}} = \mu \underbrace{dt}_{\text{infinitesimal time interval}} + \sigma\, d\underbrace{W(t)}_{\text{is the Brownian motion at time } t}$$

$$\lim_{h \to 0} \frac{S(t+h) - S(t)}{h} = \text{variation of the stock price in a "tiny/tiny.../interval"}$$

$\frac{dS(t)}{S(t)} \rightarrow$ related with log-returns.

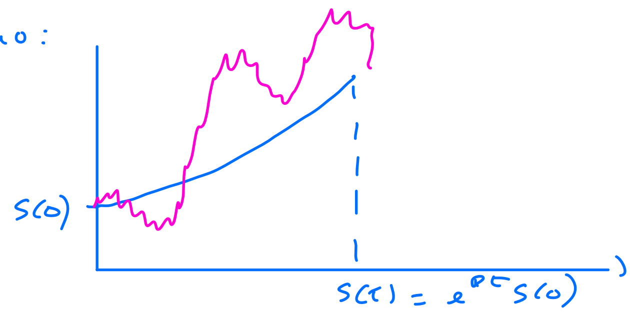
So we look closely at equation $(*)$

$$dS(t) = \underbrace{\mu S(t)\,dt}_{\text{drift}} + \underbrace{\sigma S(t)\,dW(t)}_{\text{volatility}}$$

deterministic — Random part

volatility is zero:

$$dS(t) = \mu S(t)\,dt$$
$$\Downarrow$$
$$S(t) = S(0)e^{Rt}$$



but volatility exists

$dS(t) = \mu S(t)\,dt$: (ordinary) differential equation

$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t)$: stochastic differential equation (SDE)

This particular equation has a known solution, and it is called the Geometric Brownian Motion:

$$S(t) = S(0)\,e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)}$$

can be proved (and we will prove!) that is solution of the above SDE.

<u>Definition</u>: An Itô's diffusion is a stochastic process $\{X(t), t \geq 0\}$, such that it is solution of a stochastic differential equation as follows:

$$(*) \quad \boxed{dX(t) = a(t, X(t))\,dt + b(t, X(t))\,dW(t)}$$

Particular example of the geometric Brownian motion is:

$$a(t, x(t)) = \mu x(t)$$
$$b(t, x(t)) = \sigma x(t)$$

note: That are some conditions about the functions a and b but here we will skip those conditions!

Problem: According to the definition of Brownian motion, $dw(t)$ does not sense because the set of points of a sample path where we may compute derivative is empty!) So what do we mean by $dw(t)$?

$$dw(t) \text{ “}=\text{” } \lim_{h \to 0} \frac{w(t+h) - w(t)}{h}$$

Instead of writing (*) as a differential form, we use the integral representation:

$$dx(t) = a(t, x(t))\,dt + b(t, x(t))\,dw(t)$$

$$\Leftrightarrow$$

$$\int_0^t dx(s) = \int_0^t a(s, x(s))\,ds + \underbrace{\int_0^t b(s, x(s))\,dw(s)}_{\boxed{\text{Itô's integral}}}$$

So this leads us to Itô's integrals. As for the conditional expectation, we will skip the construction and definition of Itô's integrals. We just need to know their properties.

An Itô's integral is any integral of the form:

$$\int_0^t f(s, X(s))\, dW(s)$$

and this is a random variable.

<u>Properties</u>

① $\{ \int_0^t f(s, X(s)) dW(s),\ t \geq 0 \}$ this is a stochastic process such that:

$$\int_0^t f(s, X(s))\, dW(s) \text{ is } \mathcal{F}_t^W\text{-measurable.}$$

(assuming that $\{X(s),\ s \geq 0\}$ is also $\{\mathcal{F}_s^W\}$ measurable)

$$\Rightarrow E\left[\int_0^t f(s, X(s))\, dW(s) \,\middle|\, \mathcal{F}_t^W \right] =$$

$$= \int_0^t f(s, X(s))\, dW(s) \quad (\text{a deterministic value})$$

② First Itô's isometry:

$$E\left[\int_0^t f(s, X(s))\, dW(s) \right] = 0, \quad \forall t$$

③ Second Itô's isometry:

$$E\left[\left(\int_0^t f(s, X(s))\, dW(s) \right)^2 \right] =$$

$$= \int_0^t E\left[f^2(s, X(s)) \right] ds$$

(and this last integral is the usual integral!)

② + ③ $\Rightarrow \operatorname{Var}\left(\int_0^t f(s, X(s))\, dW(s) \right)$

$$= \int_0^t E\left[f^2(s, X(s)) \right] ds$$

(4) $\left\{ \int_0^t f(s, X(s))\, dW(s),\ t \geq 0 \right\}$ is a martingale with respect to $\left\{ \mathcal{F}_t^W,\ t \geq 0 \right\}$

<u>Proof</u>

$$E\left[\int_0^t f(s, X(s))\, dW(s) \,\middle|\, \mathcal{F}_u^W \right] =$$

$$= \int_0^u f(s, X(s))\, dW(s), \quad \forall\ u \leq t \qquad = (*)$$

$$Y(t) = \int_0^t f(s, X(s))\, dW(s)$$

$$(*) \quad E\left[Y(t) \,\middle|\, \mathcal{F}_u^W \right] = Y(u) \qquad \forall\ u \leq t$$

$$E\left[\int_0^u f(s, X(s))\, dW(s) + \int_u^t f(s, X(s))\, dW(s) \,\middle|\, \mathcal{F}_u^W \right]$$

$$= E\left[\int_0^u f(s, X(s))\, dW(s) \,\middle|\, \mathcal{F}_u^W \right] +$$

$$+ E\left[\int_u^t \underbrace{f(s, X(s))\, dW(s)} \,\middle|\, \mathcal{F}_u^W \right]$$

is related with the Brownian motion between time u and time t

$$= \int_0^u f(s, X(s))\, dW(s) + \underbrace{E\left[\int_u^t f(s, X(s))\, dW(s) \right]}_{= 0}$$

due to the first Itô's isometry

(*) 

$$= \int_0^u f(s, X(s))\, dW(s) = Y(u) \qquad \blacksquare$$

1. Our definition of stochastic integral w.r.t. to Brownian motion is based on:

$$\int_0^t g(u)dW(u) := \sum_{k=0}^{n-1} g(t_k)\left[W(t_{k+1}) - W(t_k)\right]$$

where $\{0 = t_0, t_1, \dots, t_n = t\}$ is one partition of the interval $[0, t]$.

Suppose now that we define a new type of integral, based on the following:

$$\oint_0^t g(u)dW(u) := \sum_{k=0}^{n-1} g(t_{k+1})\left[W(t_{k+1}) - W(t_k)\right]$$

Let $g(t) = W(t_k)$, for $t \in [t_k; t_{k+1}[$.

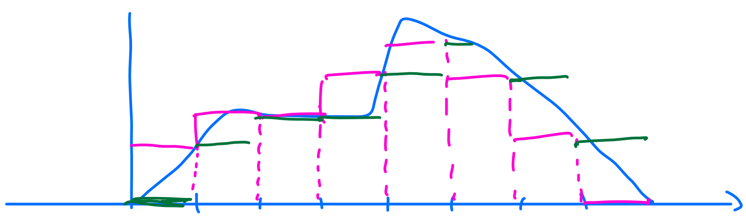
a) Show that $\mathbb{E}\left[\int_0^t g(u)dW(u)\right] = 0$ but $\mathbb{E}\left[\oint_0^t g(u)dW(u)\right] = t$.

b) Is this new integral still a martingale?

Let $\quad Y(t) = \sum_{k=0}^{n-1} W(t_k)\left[W(t_{k+1}) - W(t_k)\right]$

(in fact this is $\int_0^t W(s)\,dW(s)$)

$\quad X(t) = \sum_{k=0}^{n-1} W(t_{k+1})\left[W(t_{k+1}) - W(t_k)\right]$



$E[Y(t)] = E\left[\sum_{k=0}^{n-1} W(t_k)\left[W(t_{k+1}) - W(t_k)\right]\right]$

$= \sum_{k=0}^{n-1} E\left[W(t_k)\left(W(t_{k+1}) - W(t_k)\right)\right] = 0$

$W(t_k) \perp\!\!\!\perp \left(W(t_{k+1}) - W(t_k)\right) \Rightarrow$

$E\left[W(t_k)\left(W(t_{k+1}) - W(t_k)\right)\right] =$

$= E\left[W(t_k)\right] E\left[W(t_{k+1}) - W(t_k)\right] = 0$

$W(t_k) \sim \mathcal{N}(0, t_k) \qquad W(t_{k+1}) - W(t_k) \sim \mathcal{N}(0, \dots)$

$$E[X(t)] = E\left[\sum_{k=0}^{n-1} W(\tau_{k+1})\left(W(\tau_{k+1}) - W(\tau_k)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[W(\tau_{k+1})\left(W(\tau_{k+1}) - W(\tau_k)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[\left(\underbrace{W(\tau_{k+1}) - W(\tau_k)} + \underbrace{W(\tau_k)}\right)\underbrace{\left(W(\tau_{k+1}) - W(\tau_k)\right)}\right]$$

$$= \sum_{k=0}^{n-1} E\left[\left(W(\tau_{k+1}) - W(\tau_k)\right)^2 + \underline{W(\tau_k)\left(W(\tau_{k+1}) - W(\tau_k)\right)}\right]$$

$$= \dots = \sum_{k=0}^{n-1} E\left[\left(W(\tau_{k+1}) - W(\tau_k)\right)^2\right] + 0$$

$$= \sum_{k=0}^{n-1} \mathrm{var}\left(W(\tau_{k+1}) - W(\tau_k)\right) = \circledast$$

$$\left[\mathrm{var}(X) = E[X^2] - E^2[X]\right]$$

$$W(\tau_{k+1}) - W(\tau_k) \underset{\text{STAT. INCREMENTS}}{\sim} \mathcal{N}(0, \tau_{k+1} - \tau_k)$$

$$\left(W(t) \sim \mathcal{N}(0, t)\right)$$

$$\circledast = \sum_{k=0}^{n-1} (\tau_{k+1} - \tau_k) = \tau_{n-1+1} - \tau_0 = \tau_n - \tau_0 = t - 0 = t$$

b) Is $\left\{\sum_{k=0}^{n-1} W(\tau_{k+1})\left(W(\tau_{k+1}) - W(\tau_k)\right), n \geq 0\right\}$

a martingale?

$$E\left[\sum_{k=0}^{n} W(t_{k+1})\left(W(t_{n+1}) - W(t_n)\right) \middle| \mathcal{F}_{n-1}^{W}\right]$$

$$\stackrel{?}{=} \sum_{k=0}^{n-1} W(\tau_{k+1})\big(W(\tau_{k+1})-W(\tau_k)\big) \qquad W_1, W_2, W_3 \ldots W_{n-1}$$

$$E\Big[\sum_{k=0}^{n-1} W(\tau_{k+1})\big(W(\tau_{k+1})-W(\tau_k)\big) +$$

$$+ W(t_{n+1})\big(W(\tau_{n+1})-W(\tau_n)\big) \,\Big|\, \mathcal{F}^W_{n-1}\Big]$$

$$= E\Big[\sum_{k=0}^{n-2} W(\tau_{k+1})\big(W(\tau_{k+1})-W(\tau_k)\big) +$$

$$+ W(\tau_n)\big(W(\tau_n)-W(\tau_{n-1})\big) + W(t_{n+1})\big(W(\tau_{n+1})-W(\tau_n)\big) \,\Big|\, \mathcal{F}^W_{n-1}\Big]$$

$$= \sum_{k=0}^{n-2} W(\tau_{k+1})\big(W(\tau_{k+1})-W(\tau_k)\big) +$$

$$+ \overbrace{E\Big[W(t_n)\big(W(t_n)-W(\tau_{n-1})\big) \,\Big|\, \mathcal{F}^W_{n-1}\Big]}^{b}$$

$$+ E\Big[W(t_{n+1})\big(W(\tau_{n+1})-W(\tau_n)\big) \,\Big|\, \mathcal{F}^W_{n-1}\Big]$$

$$= A + b + c$$

$$b: \quad E\Big[\big(W(\tau_n)-W(\tau_{n-1})+W(\tau_{n-1})\big)\big(W(\tau_n)-W(\tau_{n-1})\big) \,\Big|\, \mathcal{F}^W_{n-1}\Big]$$

$$= E\Big[\big(W(\tau_n)-W(\tau_{n-1})\big)^2 + W(\tau_{n-1})\big(W(\tau_n)-W(\tau_{n-1})\big) \,\Big|\, \mathcal{F}^W_{n-1}\Big]$$

$$= E\Big[\big(W(\tau_n)-W(\tau_{n-1})\big)^2\Big] + W(\tau_{n-1})\, E\Big[W(\tau_n)-W(\tau_{n-1}) \,\Big|\, \mathcal{F}^W_{n-1}\Big]$$

$$= \operatorname{var}\big(W(\tau_n)-W(\tau_{n-1})\big) + W(\tau_{n-1})\, E\big(\underbrace{\cancel{W(\tau_n)-W(\tau_{n-1})}}_{0}\big)$$

$$= (t_n - t_{n-1})$$

∎

Because the properties of Itô's integral are different from the properties of usual integrals, also we will have different ways to define derivatives of functions of the Brownian motion.

Example: $\{S(t), t \geq 0\}$ is the stock price such that

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t)$$

what about

$$d(e^{-Rt} S(t)) = ?$$

Example

usual derivative of product:

$$d[f(x)g(x)] = f(x)\,dg(x) + df(x)\,g(x)$$

If instead, f and g depend on $\{W(t)\}$:

$$d[f(W(t))g(W(t))] = f(W(t))\,dg(W(t)) + g(W(t))\,df(W(t)) + df(W(t))\,dg(W(t))$$